Remarks on Invariant Subspaces for Finite Dimensional Operators

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ABSTRACT

Every invariant subspace of the commutant $\{A\}'$ of an operator A is the range of some operator in $\{A\}'$. If two operators have the same lattice of invariant subspaces, then each is similar to a polynomial in the other.

Let $\mathcal{H}$ be a finite dimensional (complex) vector space of dimension n. We denote the subspace generated by $x_1, x_2, \dots,$ vectors in $\mathcal{H}$, by $[x_1, x_2, \dots]$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is an invariant subspace for an (linear) operator on $\mathcal{H}$ if $A\mathcal{M} \subseteq \mathcal{M}$; it is an invariant subspace for a set $\mathcal{S}$ of operators if it is invariant for every operator in $\mathcal{S}$. The set of invariant subspaces for an operator A, with subspace sum as join and intersection as meet, is a lattice, and is denoted by Lat A. The commutant of a set $\mathcal{S}$ of operators, denoted by $\mathcal{S}'$, is the set of operators T such that $TS = ST$ for all $S \in \mathcal{S}$.

We show here that (1) every invariant subspace for the commutant $\{A\}'$ of a single operator A is the range of some operator in $\{A\}'$; (2) if two operators have the same lattice of invariant subspaces, then each of them is similar to a polynomial in the other. We begin with the following lemma.

LEMMA 1. *If N is a nilpotent operator on $\mathcal{H}$ (n-dimensional) with a cyclic vector x, then every invariant subspace for N is the range of a power of N.*

Proof. The set $\{x, Nx, \dots, N^{n-1}x\}$ is a basis of the space $\mathcal{H}$, and Lat $N = \{0, [N^{n-1}x], [N^{n-2}x, N^{n-1}x], \dots, \mathcal{H}\}$. For each $j = 1, 2, \dots, n$

$$[N^{n-1}x, N^{n-2}, \dots, N^{n-j}x] = (N^{n-j})\mathcal{H}. \qquad \blacksquare$$

PROPOSITION 2. *If A is an operator on $\mathcal{H}$, then any invariant subspace for $\{A\}'$ is the range of some operator in $\{A\}'$.*

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Proof. Let $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2\oplus\cdots\oplus\mathcal{H}_k$ be the Jordan decomposition of $\mathcal{H}$ with respect to A, so that each $\mathcal{H}_j$ reduces A and the restriction of A to $\mathcal{H}_j$ is a single Jordan cell [2]. Then

$$A=A_1\oplus A_2\oplus\cdots\oplus A_k,$$

and each A_j is of the form $\lambda_j I_j+N_j$, where I_j denotes the identity operator on $\mathcal{H}_j$ and N_j is a nilpotent operator on $\mathcal{H}_j$ with a cyclic vector. Let $\mathfrak{M}$ be an invariant subspace for $\{A\}'$, and E_j the idempotent projecting $\mathcal{H}$ onto $\mathcal{H}_j$ along $\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_{j-1}\oplus\mathcal{H}_{j-1}\oplus\mathcal{H}_{j+1}\oplus\cdots\oplus\mathcal{H}_k$. Then $E_j\mathfrak{M}$ is invariant under A_j, and hence under N_j. By Lemma 1, $E_j\mathfrak{M}=(N_j^{l_j})\mathcal{H}_j$ for some positive integer l_j. Since

$$N_jN_i=0=N_iN_j \qquad \text{if} \quad i\neq j,$$

we have

$$\begin{aligned}\mathfrak{M}&=E_1\mathfrak{M}\oplus E_2\mathfrak{M}\oplus\cdots\oplus E_k\mathfrak{M}=N_1^{l_1}\mathcal{H}_1\oplus N_2^{l_2}\mathcal{H}_2\oplus\cdots\oplus N_k^{l_k}\mathcal{H}_k\\ &=\left(N_1^{l_1}\oplus\cdots\oplus N_k^{l_k}\right)\mathcal{H}.\end{aligned}$$

Also, $N_1^{l_1}\oplus\cdots\oplus N_k^{l_k}$ commutes with A, since each N_j does. This completes the proof. ∎

In [1] Brickman and Fillmore prove that if A and B are commuting operators on a finite dimensional space and every invariant subspace of A is invariant under B, then B is a polynomial in A. The following natural question was raised by Fillmore (through a conversation): what is the conclusion if the commutativity assumption is dropped and the inclusion of invariant subspace lattices is strengthened to equality?

The following example shows that B can have exactly the same lattice of invariant subspaces as A without being a polynomial in A.

EXAMPLE 3. On the space $\mathbf{C}^3$, let A and B be defined, respectively, by the matrices

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\quad\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}$$

with respect to the standard orthonormal ordered basis $\{e_1,e_2,e_3\}$. Then A

and B have exactly the same lattice of invariant subspaces, viz. $\{\{0\}, [e_1], [e_1, e_2], [e_1, e_2, e_3] = \mathbf{C}^3\}$, while neither is a polynomial in the other. (In fact A and B do not commute.) But B is similar to A. The following result shows that this is the general situation.

PROPOSITION 4. *If A and B are operators on the n-dimensional space $\mathcal{H}$, with exactly the same lattice of invariant subspaces, then each is similar to a polynomial in the other.*

Proof. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$ be the Jordan decomposition of $\mathcal{H}$ with respect to A as in the proof of the previous proposition. Then $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, and by the hypothesis, each $\mathcal{H}_i$ reduces B, so $B = B_1 \oplus B_2 \oplus \cdots \oplus B_k$, where B_i is the restriction $B|\mathcal{H}_i$. Each A_i is a nilpotent operator with a cyclic vector plus a scalar, B_i has exactly the same lattice of invariant subspaces as A_i, and hence each B_i is also a nilpotent operator with a cyclic vector plus a scalar. Therefore, $A_i = \lambda_i E_i + S_i^{-1} B_i S_i$, where S_i is an invertible operator leaving invariant every invariant subspace of A_i, and E_i is the identity operator on $\mathcal{H}_i$ (or the idempotent projecting $\mathcal{H}$ onto $\mathcal{H}_i$ along $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{i-1} \oplus \mathcal{H}_{i+1} \oplus \cdots \oplus \mathcal{H}_k$). Let $S = S_1 \oplus S_2 \oplus \cdots \oplus S_k$. Then

$$SE_i = E_i S = S_i, \qquad S^{-1}BS + \sum_{i=1}^{k} \lambda_i E_i = A.$$

It is easy to see that A_i and A_j have the same eigenvalue if and only if B_i and B_j do. Since each sum of the E_i's, with A_i's having the same eigenvalue, is a polynomial in A [2], and S commutes with each E_i, we obtain $S^{-1}BS = p(A)$ for some polynomial p. The proof is complete. ■

REFERENCES

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2 K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, 1962.

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